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Embedding of Weak Markov Systems

R. A. ZALIK

Department of Mathematics, Auburn University, Auburn, Alabama 36849, U.S.A.

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We shall adopt the following nomenclature: let A be a subset of the real line having at least $n+2$ elements ($n \geqslant 0$), let I be the convex hull of A and let $Z_n = \{z_0,..., z_n\}$ be a sequence of linearly independent real valued functions defined on A; then Z_n is called a (weak) Čebyšev system on A if for every choice of $n+1$ points t_i of A with $t_0 < t_1 < \cdots < t_n$, $\det[z_i(t_j)] > 0$ ($\geqslant 0$). If $\{z_0,..., z_i\}$ is a (weak) Čebyšev system for $i = 0,..., n$, then Z_n will be called a (weak) Markov system. A normed (weak) Markov system is a (weak) Markov system Z_n for which $z_0 \equiv 1$. Markov systems are also called complete Čebyšev systems or CT-systems (cf. Karlin & Studden [2]). If every element of Z_n is bounded in the intersection of A with any compact subset of I, we shall say that Z_n is C-bounded on A.

Not every weak normed Markov system is C-bounded. For example, let the functions u_i be defined as follows: for $-1 < x < 0$, $u_0(x) = 1$, $u_1(x) = u_2(x) = 0$; for $0 < x < 1$, $u_0(x) = u_1(x) = 1$, $u_2(x) = \ln x$; then $\{u_0, u_1, u_2\}$ is a normed weak Markov system on $(-1, 0) \cup (0, 1)$ but u_2 is unbounded in every set of the form $[\alpha, 0) \cup (0, \beta]$, where $-1 < \alpha < 0 < \beta < 1$.

If $U_n = \{u_0,..., u_n\}$ is a set of real valued functions defined on a real set A and $V_n = \{v_0,..., v_n\}$ is a set of real valued functions defined on a real set B we say that U_n can be embedded in V_n if there is a strictly increasing function $h : A \to B$ such that $v_i[h(t)] = u_i(t)$ for every t in A and $i = 0,..., n$. The function h is called an embedding function. We have:

THEOREM. *A normed weak Markov system U_n on a set A can be embedded in a normed weak Markov system of continuous functions defined on an open bounded interval if and only if U_n is C-bounded on A. Moreover if c is an arbitrary element of A, the embedding function h can be chosen so that $h(c) = c$.*

Remarks. (1) A similar result for Čebyšev systems was proved by Gopinath and Kurshan in [1, Theorem 3.1].

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(2) Stockenberg [4] has shown that if U_n is a weak Čebyšev system on A and A has no smallest nor largest element, then the linear span of U_n contains a basis that is a weak Markov system on A (cf. [4, Theorem 3]). An analogous theorem for Čebyšev systems was obtained by the author in [6] (for other proofs see Stockenberg [5], and Gopinath and Kurshan [1]).

Proof of Theorem. Let $U_n = \{u_0, \ldots, u_n\}$ be a C-bounded weak Markov system defined on a set A, and let $l_1 = \inf(A)$, $l_2 = \sup(A)$. If for instance l_1 is in A, let u_i^* coincide with u_i on A and equal $u_i(l_1)$ on $(-\infty, l_1)$. It is clear that $U_n^* = \{u_0^*, \ldots, u_n^*\}$ is a normed weak Markov system and that U_n can be embedded in U_n^*, with $h(t) = t$ as the embedding function. It is therefore clear that there is no loss of generality in assuming that neither l_1 nor l_2 belong to A. Let A^c denote the closure of A in the relative topology of (l_1, l_2). Define y_i on A^c as follows: $y_i(t) = u_i(t)$ on A, and if t is a point of accumulation of A that does not belong to A $y_i(t) = \limsup_{x \to t} u_i(x)$. From the hypotheses we know that the functions y_i are well defined. Clearly $\{y_0, \ldots, y_n\}$ is a C-bounded weak normed Markov system and U_n can be embedded in it.

In view of the preceding remarks, there is no loss of generality in assuming that U_n is defined on a set A such that neither l_1 nor l_2 belong to it and such that A is closed in the relative topology of (l_1, l_2). With these assumptions the complementary set of A in (l_1, l_2) is a disjoint union of open intervals V_j; moreover, if $c_j = \inf(V_j)$, it is clear that c_j belongs to A. Let $\bar{u}_i(t)$ be defined in (l_1, l_2) as follows: $\bar{u}_i(t) = u_i(t)$ on A, and for each j, $\bar{u}_i(t) = u_i(c_j)$ on V_j. Clearly U_n can be embedded in $\bar{U}_n = \{\bar{u}_0, \ldots, \bar{u}_n\}$. Moreover, it is easy to see that $\bar{U}_n$ is a normed weak Markov system on (l_1, l_2): assume, for instance, that $t_0 < \cdots < t_k$, $(k \leqslant n)$, that all the t_j except for t_r are in A, and that t_r is in V_m for some m. Defining $x_j = t_j$ if $j \neq r$ and $x_r = c_m$ it is clear that $x_0 < \cdots < x_{r-1} \leqslant x_r < \cdots < x_k$, and that all the points x_j are in A. Thus $\det|\bar{u}_i(t_j)| = \det|u_i(x_j)| \geqslant 0$.

The discussion of the preceding paragraphs shows that every C-bounded normed weak Markov system can be embedded in a C-bounded normed weak Markov system defined in an open interval. Thus, in the sequel we shall assume that U_n is defined on an open interval $I = (a, b)$ and is C-bounded thereon. From [7, Lemma 4.1; 3, Theorem 6] we readily conclude that the functions u_i are of bounded variation in every closed subinterval of I.

Assume that the functions $u_1(t), \ldots, u_r(t)$ are continuous on I and let $\{t_j\}$ denote the set of points of discontinuity of $u_{r+1}(t)$. Let $\alpha_j = |u_{r+1}(t_j^+) - u_{r+1}(t_j)|$, $\beta_j = |u_{r+1}(t_j) - u_{r+1}(t_j^-)|$. Let $h(t)$ be defined as follows: if $t \notin \{t_j\}$, $h(t) = t + \sum_{t_j < t} (\alpha_j + \beta_j)$, whereas $h(t_i) = t_i + \sum_{t_j < t_i} (\alpha_j + \beta_j) + \alpha_i$.

Clearly h is strictly increasing and if $a_1 = h(a^+)$, $b_1 = h(b^-)$, $h(I)$ is contained in (a_1, b_1). Let C denote the complementary set of $h(I)$ in (a_1, b_1).

Then

$$C = \bigcup \{h(t_i^-), h(t_i^-) + \alpha_i) \cup (h(t_i^-) + \alpha_i, h(t_i^-) + \alpha_i + \beta_i]\},$$

where it is understood that $[x, x) = (x, x] = \varnothing$.

Let $w_i(t)$ be defined on (a_1, b_1) as follows: if t belongs to $h(I)$, $w_i(t) = u_i[h^{-1}(t)]$, whereas if t belongs to C $w_i(t)$ is defined by linear interpolation; for instance on $[h(t_j^-), h(t_j^-) + \alpha_j)$,

$$w_i(t) = \alpha_j^{-1}[h(t_j^-) + \alpha_j - t]\, u_i(t_j^-) + \alpha_j^{-1}[t - h(t_j^-)]\, u_i(t_j).$$

It is clear that $h(t)$ embeds U_n in W_n and that the functions w_i, $i = 1,..., r + 1$, are continuous on (a_1, b_1). It is also easy to see that W_n is a normed weak Markov system on (a_1, b_1): Let $s \leqslant n$ and $x_0 < \cdots < x_s$, and assume, for example, that for some m and j, x_m is in $[h(t_j^-), h(t_j^-) + \alpha_j)$ and that all other x_k are in $h(I)$. If $v_i(t_j) = u_i(t_j^-)$ and $v_i(t) = u_i(t)$ elsewhere in I, it is clear that $\{v_0,..., v_n\}$ is a normed weak Markov system on I. Let $s_m = t_j$ and for $k \neq m$, $s_k = h^{-1}(x_k)$; then $s_0 < \cdots < s_{m-1} \leqslant s_m < s_{m+1} < \cdots < s_m$ and we have

$$\begin{aligned} \det|w_i(x_k)| = {} & \alpha_j^{-1}[h(t_j^-) + \alpha_j - x_m]\det[v_i(s_k)] \\ & + \alpha_j^{-1}[x_m - h(t_j^-)]\det[u_i(s_k)] \geqslant 0. \end{aligned}$$

Making, if necessary, an arctan change of variable, we can assume that (a_1, b_1) is a bounded interval.

Repeating a finite number of times the procedure described in the preceding paragraph, we infer that there is a bounded interval (α, β) and a normed weak Markov system V_n of continuous functions on (α, β) such that U_n can be embedded in V_n. Let $q(t)$ be the embedding function, and let c be an arbitrary point in the domain of the functions u_i. Defining $q_1(t) = q(t) - q(c) + c$ and $v_i^*(t) = v_i(t + q(c) - c)$ it is clear that $q_1(c) = c$ that V_n^* is a continuous normed weak Markov system on an open interval and that $q_1(t)$ embeds U_n in V_n^*, whence the conclusion follows. The proof of the converse is trivial and will be omitted. Q.E.D.

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